WEEKLY PROBLEM NOVEMBER 8 TO NOVEMBER 14 2009

RATIONAL POINTS ON LINES

A *rational* number is a quotient of two integers a and b. We write rational numbers as *fractions*

$$\frac{a}{b}$$

A *line* in the (x, y)-plane is described by a polynomial of the form

$$a \cdot x + b \cdot y + c = 0$$

Where a, b, c are real numbers. Why is this form of a line equivalent to the *slope intercept* form of a line that one learns in algebra I? A rational point in the (x, y)-plane is a point (a, b) where a and b are rational numbers. Similarly, an integral point in the plane is a point (a, b) where a and b are integers. For example

$(1, -2)$ is both a rational point and an integral point

$\left(\frac{1}{7}, -\frac{4}{3}\right)$ is a rational point but not an integral point

$\left(\sqrt{2}, 1\right)$ is neither a rational point nor an integral point

A line of the form $ax + by + c = 0$ is called a rational line precisely when a, b, c are all rational numbers. The same line is called integral precisely when a, b, c are all integers. How do a, b, c relate to the points (x, y) that lie on the line? Do integral lines only pass through integral points? Do rational lines only pass through rational points? More precisely, are the following statements true? Explain why or why not.

$$ax + by + c = 0 \text{ is an integral line} \overset{?}{\iff} \text{the only points } (x, y) \text{ on the line are integral points.}$$

$$ax + by + c = 0 \text{ is a rational point} \overset{?}{\iff} \text{the only points } (x, y) \text{ on the line are rational points.}$$

Given an integral line $ax + by + c = 0$ can you determine all the integral points (x, y) that are on the line (i.e. that satisfy the equation $ax + by + c = 0$) in terms of a, b, c? What about the rational points, can you characterize all rational points satisfying $ax + by + c = 0$ in terms of a, b, c? Can you answer the same questions if you change the line from an integral line to a rational line?

What about the intersection of two integral lines, must they intersect at an integral point? Must they intersect at a rational point? Lastly, can you construct an integral line that passes through exactly one integral point? Can you construct a rational line that passes through exactly one rational point?